1. Consider the subspaces:

$$\mathrm{U} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathrm{V} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

a. Determine whether U and V are complementary subspaces.

solution:

U and V will be complementary if and only if the combination (union) of their bases is a basis for $\Re^3$. In this case, the combined bases form the columns of

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

But direct computation (i.e. eliminating the augmented matrix $[\ \mathbf{B} \mid \mathbf{I}\ ]$ to row-reduced echelon form) shows that

$$\mathbf{B}^{-1} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Therefore, since $\mathbf{B}$ is invertible, its columns must be a basis for $\Re^3$. Hence, U and V are complementary subspaces.

b. Determine whether U and V are complementary orthogonal subspaces.

solution:

U and V will be complementary orthogonal subspaces if and only if they are complementary, and if $\mathbf{V}^H\mathbf{U} = \mathbf{0}$, where $\mathbf{U}$ and $\mathbf{V}$ are matrices whose columns are bases for U and V, respectively. In this case

$$\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{V}^H\mathbf{U} = [\ 0\ \ 2\ \ 1\ ] \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = [\ 1\ \ 4\ ] \neq \mathbf{0}$$

and therefore, in this case U and V are **not** orthogonal complements.

c. Find the matrix for the projector onto U along V , using the standard representation:

$$\mathbf{P} = \mathbf{B}\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{B}^{-1}.$$

where $\mathbf{B}$ is the matrix whose columns represent, sequentially, bases for U and V .

solution:

We have already found $\mathbf{B}$ and $\mathbf{B}^{-1}$ above. So, since dim(U)=2, we have:

$$\mathbf{P} = \mathbf{B}\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{B}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

or

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

d. If U and V are orthogonal complements, find the matrix for the projection onto U using the representation:

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$$

where $\mathbf{A}$ is the matrix whose columns are a basis for U and compare that the result from part c. above.

solution:

Since, in this case, U and V are not orthogonal complements, then this part does not apply.

e. For the matrix $\mathbf{P}$ found in part c., show by direct computation that $\mathbf{P}^2 = \mathbf{P}$.

solution:

Direct computation (e.g. MATLAB) shows that

$$\mathbf{P}^2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix} = \mathbf{P}$$

f. For the vector $\mathbf{x} = [\ 2\ \ 3\ \ 1\ ]^T$, find the projector of $\mathbf{x}$ onto U along V :

(1.) Using $\mathbf{P}$ as determined in part c. above.

solution:

Using $\mathbf{P}$ as determined in part c., then by definition, the projector of $\mathbf{x}$ onto U along V is:

$$\mathbf{Px} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$$

(2.) By finding the coordinates of $\mathbf{x}$ in terms of the basis for U and V , and compare the two.

solution:

By definition, the coordinates of $\mathbf{x}$ in terms of the basis for U and V are the solution of $\mathbf{B}\,[\mathbf{x}]_{\mathbf{B}} = \mathbf{x}$, where $\mathbf{B}$ is the matrix whose columns, sequentially, represent the basis for U and V , i.e., in this case

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad \Longrightarrow \quad [\mathbf{x}]_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{x} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}$$

But, in this case, only the first two columns of $\mathbf{B}$ correspond to the basis for U , and therefore the component of $\mathbf{x}$ in U is precisely the result of applying only the first two of these coordinates to that basis, i.e.

$$\mathbf{P}\,\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$$

i.e., exactly the same result as in part (1.) above. (Which, of course, is precisely what should be expected.

g. Compare the projector of $\mathbf{x}$ onto U along V computed in part f. above with the original vector $\mathbf{x}$ and explain any differences.

solution:

As found in part f.,

$$\mathbf{P}\,\mathbf{x} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \quad \text{while} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Obviously, these are not identical. That should not be unexpected, since $\mathbf{P}\,\mathbf{x}$ only represents the component of $\mathbf{x}$ in U , which, unless $\mathbf{x}$ lies entirely in U to begin with, will not be identical with $\mathbf{x}$.

h. Repeat parts f. and g. for the vector $\mathbf{y} = [\,2 \;\; -1 \;\;\; 1]^T$.

solution:

Using $\mathbf{P}$ as determined in part c., then by definition, the projector of $\mathbf{y}$ onto U along V is:

$$\mathbf{P}\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \mathbf{y}$$

In this case, the projector is identical with the original vector. Why this occurs becomes clear when we compute the coordinates of this new vector in terms of the bases for U and V ($[\mathbf{y}]_{\mathbf{B}}$), i.e.

$$[\mathbf{y}]_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{y} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

This clearly indicates that $\mathbf{y}$ has **no** component in V , and so $\mathbf{y} \in$ U, and therefore we expect $\mathbf{P}\mathbf{y} = \mathbf{y}$, which is precisely what happens.

2. Consider the subspaces:

$$\mathrm{U} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathrm{V} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and the vectors $\mathbf{x} = [\, 0 \;\; 1 \;\; 4 \,]^T$ and $\mathbf{y} = [\, 2 \;\; -1 \;\; 1 \,]^T$.

a. Determine whether U and V are complementary subspaces.

solution:

Since U and V are subspaces of $\Re^3$, they will be complementary if and only if the combination (union) of their bases is a basis for $\Re^3$. In this case, the combined bases are simply the columns of

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

But direct computation (i.e. eliminating the augmented matrix $[\,\mathbf{B} \mid \mathbf{I}\,]$ to row-reduced echelon form) shows that

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Therefore, since $\mathbf{B}$ is invertible, its columns must be a basis for $\Re^3$. Hence, U and V are complementary subspaces.

b. Determine whether U and V are complementary orthogonal subspaces.

solution:

U and V will be complementary orthogonal subspaces if and only if they are complementary, and if $\mathbf{V}^H\mathbf{U} = \mathbf{0}$, where $\mathbf{U}$ and $\mathbf{V}$ are matrices whose columns are bases for U and V , respectively. In this case:

$$\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{V}^H\mathbf{U} = [\, 1 \;\; 1 \; -1 \,] \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = [\, 0 \;\; 0 \,] = \mathbf{0}$$

solution:

Therefore, U and V are orthogonal subspaces. But, since we've already shown they are complementary, then they must be orthogonal complements.

c. Find the matrix for the projector onto U along V , using the standard representation:

$$\mathbf{P} = \mathbf{B}\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{B}^{-1}.$$

where $\mathbf{B}$ is the matrix whose columns represent, sequentially, bases for U and V .

solution:

We have already found $\mathbf{B}$ and $\mathbf{B}^{-1}$ above. So, noting that dim(U)=2, we have:

$$\mathbf{P} = \mathbf{B}\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{B}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

or

$$\mathbf{P} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

d. If U and V are orthogonal complements, find the matrix for the projection onto U using the representation:

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$$

where $\mathbf{A}$ is the matrix whose columns are a basis for U and compare that to the result from part c. above.

solution:

In this case, U and V are orthogonal complements, and therefore a matrix whose columns are a basis for U is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \implies \mathbf{A}^H \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

and so

$$\left(\mathbf{A}^H \mathbf{A}\right)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{bmatrix}$$

Therefore, either by direct computation, or using MATLAB:

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^H \mathbf{A}\right)^{-1} \mathbf{A}^H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

This, of course, is identical to $\mathbf{P}$ as determined in part c.

e. For the matrix $\mathbf{P}$ found in part c., show by direct computation that $\mathbf{P}^2 = \mathbf{P}$.

solution:

Direct computation (e.g. MATLAB) does show that

$$\mathbf{P}^2 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \mathbf{P}$$

f. For the vector $\mathbf{x} = [\ 0\ \ 1\ \ 4\ ]^T$, find the projector of $\mathbf{x}$ onto U along V :

(1.) Using $\mathbf{P}$ as determined in part c. above.

solution:

Using $\mathbf{P}$ as determined in part c., then by definition, the projector of $\mathbf{x}$ onto U along V of is:

$$\mathbf{P}\mathbf{x} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(2.) By finding the coordinates of $\mathbf{x}$ in terms of the basis for U and V , and compare the two.

solution:

By definition, the coordinates of $\mathbf{x}$ in terms of the basis for U and V are the solution of $\mathbf{B}\,[\mathbf{x}]_{\mathbf{B}} = \mathbf{x}$, where $\mathbf{B}$ is the matrix whose columns, sequentially, represent the basis for U and V , i.e., in this case

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \quad \Longrightarrow \quad [\mathbf{x}]_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{x} = \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

But, in this case, only the first two columns of $\mathbf{B}$ correspond to the basis for U , and therefore the component of $\mathbf{x}$ in U is precisely the result of applying only the first two of these coordinates to that basis, i.e.

$$\mathbf{P}\,\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e., exactly the same result as in part (1.) above. (Which, of course, is precisely what should be expected.)

g. Compare the projector of $\mathbf{x}$ onto U along V computed in part f. above with the original vector $\mathbf{x}$ and explain any differences.

solution:

As found in part f.,

$$\mathbf{P}\,\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{while} \quad \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

Obviously, these are not identical. This, however, should not be totally unexpected, since $\mathbf{P}\,\mathbf{x}$ only represents the component of $\mathbf{x}$ in U , which, unless $\mathbf{x}$ lies entirely in U to begin with, will not be identical with $\mathbf{x}$.

h. Repeat parts f. and g. for the vector $\mathbf{y} = [\;2 \;\; -1 \quad 1\;]^T$.

solution:

Using $\mathbf{P}$ as determined in part c., then by definition, the projector onto U along V of $\mathbf{y}$ is:

$$\mathbf{Py} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \mathbf{y} \quad ???$$

In this case, the projector is identical with the original vector. Why this occurs becomes clear when we compute the coordinates of this new vector in terms of the bases for U and V ($[\mathbf{y}]_{\mathbf{B}}$), i.e.

$$[\mathbf{y}]_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{y} = \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

This clearly indicates that $\mathbf{y}$ has **no** component in V , and so $\mathbf{y} \in$ U, and therefore we expect $\mathbf{Py} = y$, which is precisely what happens.

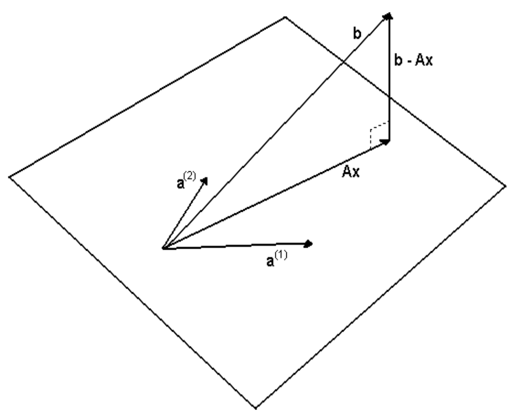
3. Find the matrix $\mathbf{P}$ which projects an arbitrary vector in $\Re^5$ onto the subspace spanned by:

$$\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Show directly that $\text{Col}(\mathbf{P})$ is identical to the span of $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$. Also show directly that $\mathbf{P}^2 = \mathbf{P}$.

solution:

The problem is best described by the following figure, which describes the analogous problem in $\Re^3$:



In terms of this problem, $\mathbf{A}$ is the matrix whose columns are the basis vectors for the given subspace and $\mathbf{A}\,\mathbf{x}$ is the desired projection, where, from the least squares formulation, we know:

$$\begin{aligned} \mathbf{A}^T \mathbf{A}\,\mathbf{x} &= \mathbf{A}^T \mathbf{b} \\ \implies \quad \mathbf{x} &= \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b} \\ \implies \quad \mathbf{A}\,\mathbf{x} &= \underbrace{\mathbf{A}\left(\mathbf{A}^T \mathbf{A}\right)^{1} \mathbf{A}^T}_{\text{projection matrix}} \mathbf{b} \end{aligned}$$

Therefore:

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T$$

solution:

or, in terms of this problem:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Direct computation (e.g. MATLAB) shows:

$$\mathbf{P} = \begin{bmatrix} \frac{3}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{2}{5} & -\frac{1}{4} & \frac{1}{20} & \frac{1}{20} & \frac{7}{20} \end{bmatrix}$$

Gaussian elimination applied to $\mathbf{P}$ yields:

$$\mathbf{U} = \begin{bmatrix} \frac{3}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and therefore only the first two columns of $\mathbf{P}$ are linearly independent, i.e. $\text{Col}(\mathbf{P})$ is two-dimensional.

To show that $\text{Col}(\mathbf{P})$ is identical to $\text{Col}(\mathbf{A})$, i.e. to the span of $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$, note that:

$$\mathbf{P} = \mathbf{A} \left(\mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T = \mathbf{A} \left\{ \underbrace{\left(\mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T}_{\mathbf{B}} \right\} = \mathbf{A}\, \mathbf{B}$$

and therefore every column of $\mathbf{P}$ is a linear combination of the columns of $\mathbf{A}$. So the span of the columns of $\mathbf{P}$ is a subspace of the span of the columns of $\mathbf{A}$. But since both are two-dimensional, they must be identical.

solution:

Alternatively, form the matrix:

$$\overline{\mathbf{AP}} = \left[\ \mathbf{A} \ \vdots \ \mathbf{P}(:,1:2)\ \right] = \begin{bmatrix} 1 & 1 & \frac{3}{5} & 0 \\ 2 & -1 & 0 & \frac{3}{4} \\ 1 & 0 & \frac{1}{5} & \frac{1}{4} \\ 1 & 0 & \frac{1}{5} & \frac{1}{4} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{4} \end{bmatrix}$$

and use Gaussian elimination to show that it reduces to:

$$\begin{bmatrix} 1 & 1 & \frac{3}{5} & 0 \\ 0 & -3 & -\frac{6}{5} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and therefore the third and fourth columns of $\overline{\mathbf{AP}}$ must be linearly dependent on the first two. But since the third and fourth columns of $\overline{\mathbf{AP}}$ are precisely the basis for Col($\mathbf{P}$), we again have that the the column spaces of $\mathbf{A}$ and $\mathbf{P}$ must be identical.

Direct (MATLAB) computation shows that

$$\mathbf{P}^2 = \begin{bmatrix} \frac{3}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{2}{5} & -\frac{1}{4} & \frac{1}{20} & \frac{1}{20} & \frac{7}{20} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{2}{5} & -\frac{1}{4} & \frac{1}{20} & \frac{1}{20} & \frac{7}{20} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{1}{5} & \frac{1}{4} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ \frac{2}{5} & -\frac{1}{4} & \frac{1}{20} & \frac{1}{20} & \frac{7}{20} \end{bmatrix} = \mathbf{P}$$

(Of course, this relationship is, essentially by definition, true for any projection!)

4. Use the Gram-Schmidt method to produce an **orthonormal** basis for the column space of

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

solution:

Note that for **hand** calculation, it's probably better to wait until you've found a full set of orthogonal vectors before you normalize. (Avoids lots of nasty square roots in the calculations.) But we'll follow the classic algorthim:

The classic Gram-Schmidt algorithm is:

(1) Form: $\mathbf{q}^{(1)} = \mathbf{a}^{(1)}/\| \mathbf{a}^{(1)} \|$

(2) For $j = 2, \ldots, n$, let

$$\mathbf{v}^{(j)} = \mathbf{a}^{(j)} - \alpha_1 \mathbf{q}^{(1)} - \cdots - \alpha_{j-1} \mathbf{q}^{(j-1)}$$

where

$$\alpha_i = \mathbf{a}^{(j)^H} \mathbf{q}^{(i)}$$

then let

$$\mathbf{q}^{(j)} = \mathbf{v}^{(j)}/\| \mathbf{v}^{(j)} \|$$

In this problem, $\| \mathbf{a}^{(1)} \| = \sqrt{1^2+1^2+1^2+0^2} = \sqrt{3}$, and therefore

$$\mathbf{q}^{(1)} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}$$

Then, for $j = 2$,

$$\mathbf{a}^{(2)} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

so

$$\alpha_1 = \mathbf{q}^{(1)^H} \mathbf{a}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

solution:

and so

$$\mathbf{v}^{(2)} \;=\; \mathbf{a}^{(2)} - \alpha_1 \mathbf{q}^{(1)} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - (\sqrt{3}) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Since $\| \mathbf{v}^{(2)} \| = \sqrt{3}$, then

$$\mathbf{q}^{(2)} \;=\; \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now, for $j = 3$,

$$\mathbf{a}^{(3)} = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

and so

$$\alpha_1 = {\mathbf{q}^{(1)}}^H \mathbf{a}^{(3)} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \frac{-3}{\sqrt{3}} = -\sqrt{3}$$

and

$$\alpha_2 = {\mathbf{q}^{(2)}}^H \mathbf{a}^{(3)} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

solution:

Therefore

$$\begin{aligned}
\mathbf{v}^{(3)} &= \mathbf{a}^{(3)} - \alpha_1 \mathbf{q}^{(1)} - \alpha_2 \mathbf{q}^{(2)} \\
&= \begin{bmatrix} -1 \\ 1 \\ -3 \\ 1 \end{bmatrix} - (-\sqrt{3}) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} - (\sqrt{3}) \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}
\end{aligned}$$

Therefore, since $\| \mathbf{v}^{(3)} \| = \sqrt{6}$,

$$\mathbf{q}^{(3)} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ 0 \end{bmatrix}$$

These are clearly orthogonal, i.e.

$$\mathbf{q}^{(1)^H} \mathbf{q}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} = 0$$

$$\mathbf{q}^{(1)^H} \mathbf{q}^{(3)} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ 0 \end{bmatrix} = 0$$

and

$$\mathbf{q}^{(2)^H} \mathbf{q}^{(3)} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ 0 \end{bmatrix} = 0$$

Furthermore, since we normalized the $\mathbf{q}^{(i)}$ as we constructed then, they are also, by definition, orthonormal.

5. Use the classic Gram-Schmidt method to produce a $\mathbf{Q}\,\mathbf{R}$ factorization of

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

solution:

The classic Gram-Schmidt algorithm, modified to produce a $\mathbf{Q}\,\mathbf{R}$ factorization as a byproduct of producing an orthonormal basis, is:

(1) Form: $\quad \mathbf{q}^{(1)} = r_{11}\ \mathbf{a}^{(1)} \quad \Longrightarrow \quad r_{11} = \|\,\mathbf{a}^{(1)}\,\|$

(2) For $\quad j = 2, \ldots, n$:

(i) let: $\quad \mathbf{v}^{(j)} \ = \ \mathbf{a}^{(j)} - r_{1j}\mathbf{q}^{(1)} - \cdots - r_{(j-1)j}\mathbf{q}^{(j-1)}$

where $\quad r_{i,j} = \mathbf{a}^{(j)^H}\,\mathbf{q}^{(i)}$

(ii) then: $\quad \mathbf{q}^{(j)} = r_{jj}\ \mathbf{v}^{(j)} \quad \Longrightarrow \quad r_{jj} = \|\,\mathbf{v}^{(j)}\,\|$

Therefore, in this problem

$$\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore

$$\|\,\mathbf{a}^{(1)}\,\| = 2 \quad \Longrightarrow \quad \mathbf{q}^{(1)} = \frac{1}{2}\mathbf{a}^{(1)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Then, for $\ j = 2$,

$$\mathbf{a}^{(2)} = \begin{bmatrix} -1 \\ -3 \\ -1 \\ -3 \end{bmatrix}$$

so

$$r_{12} = \mathbf{q}^{(1)^H}\,\mathbf{a}^{(2)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ -1 \\ -3 \end{bmatrix} = \frac{-8}{2} = -4$$

solution:

and so

$$\mathbf{v}^{(2)} \ = \ \mathbf{a}^{(2)} - r_{12}\mathbf{q}^{(1)} = \begin{bmatrix} -1 \\ -3 \\ -1 \\ -3 \end{bmatrix} - (-4) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Therefore

$$r_{22} = \| \mathbf{v}^{(2)} \| = 2 \qquad \Longrightarrow \qquad \mathbf{q}^{(2)} = \frac{1}{2}\mathbf{v}^{(2)} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Now, for $j = 3$,

$$\mathbf{a}^{(3)} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ -3 \end{bmatrix}$$

and so

$$r_{13} = \mathbf{q}^{(1)H}\,\mathbf{a}^{(3)} = [\ \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\ ] \begin{bmatrix} 5 \\ -1 \\ 3 \\ -3 \end{bmatrix} = \frac{4}{2} = 2$$

and

$$r_{23} = \mathbf{q}^{(2)H}\,\mathbf{a}^{(3)} = [\ \tfrac{1}{2} \quad -\tfrac{1}{2} \quad \tfrac{1}{2} \quad -\tfrac{1}{2}\ ] \begin{bmatrix} 5 \\ -1 \\ 3 \\ -3 \end{bmatrix} = \frac{12}{2} = 6$$

and

$$\mathbf{v}^{(3)} \ = \ \mathbf{a}^{(3)} - r_{13}\mathbf{q}^{(1)} - r_{23}\mathbf{q}^{(2)} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ -3 \end{bmatrix} - (2) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - (6) \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

solution:

Therefore

$$r_{33} = \| \mathbf{v}^{(3)} \| = 2 \qquad \Longrightarrow \qquad \mathbf{q}^{(3)} = \frac{1}{2}\mathbf{v}^{(3)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

The fact that these are orthogonormal, i.e. that ${\mathbf{q}^{(i)}}^T \mathbf{q}^{(j)} = 0 \ , \quad i \neq j$ and $\| \mathbf{q}^{(i)} \| = 1$ can easily be checked. For example

$${\mathbf{q}^{(1)}}^T \mathbf{q}^{(3)} = [\ \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\ ] \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = 0$$

and

$$\| \mathbf{q}^{(2)} \| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2} = 1$$

Therefore

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and, using the values computed above:

$$\mathbf{R} = \begin{bmatrix} 2 & -4 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Direct computation will verify that

$$\mathbf{Q}\,\mathbf{R} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

6. Use the modified Gram-Schmidt method to produce a $\mathbf{Q\,R}$ factorization of

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

solution:

The modified Gram-Schmidt algorithm, produces a $\mathbf{Q\,R}$ factorization as a byproduct of producing an orthonormal basis, according to:

(1) Let: $\mathbf{v}^{(i)} = \mathbf{a}^{(1)} \quad , i = 1, \ldots, n$

(2) For $j = 1, \ldots, n$:

Form: $\mathbf{q}^{(j)} = r_{jj}\ \mathbf{v}^{(j)} \qquad \Longrightarrow \qquad r_{jj} = \|\, \mathbf{v}^{(j)} \,\|$

Remove the component in the direction of $\mathbf{q}^{(j)}$ from each remaining $\mathbf{v}^{(k)}$ using:

$$\mathbf{v}^{(k)} = \mathbf{v}^{(k)} - r_{jk}\mathbf{q}^{(j)}$$

where $r_{jk} = \mathbf{q}^{(j)^H}\, \mathbf{v}^{(k)}$

Therefore, in this problem

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ -3 \\ -1 \\ -3 \end{bmatrix} \quad , \text{ and } \ \mathbf{v}^{(3)} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ -3 \end{bmatrix}$$

So, for $j = 1$

$$\|\, \mathbf{v}^{(1)} \,\| = 2 \quad \Longrightarrow \quad \mathbf{q}^{(1)} = \frac{1}{2}\mathbf{v}^{(1)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Now remove any components of this from the remaining columns, i.e. for $k = 2$:

$$r_{12} = \mathbf{q}^{(1)^H}\, \mathbf{v}^{(2)} = [\ \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\ ] \begin{bmatrix} -1 \\ -3 \\ -1 \\ -3 \end{bmatrix} = \frac{-8}{2} = -4$$

solution:

and so $\mathbf{v}^{(2)}$ becomes

$$\mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ -3 \\ -1 \\ -3 \end{bmatrix} - (-4) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Similarly, for $k = 3$, we have:

$$r_{13} = {\mathbf{q}^{(1)}}^H \mathbf{v}^{(3)} = [\ \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\ ] \begin{bmatrix} 5 \\ -1 \\ 3 \\ -3 \end{bmatrix} = \frac{4}{2} = 2$$

and so $\mathbf{v}^{(3)}$ becomes

$$\mathbf{v}^{(3)} \ = \ \mathbf{v}^{(3)} - r_{13}\mathbf{q}^{(1)} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ -3 \end{bmatrix} - (2) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \\ -4 \end{bmatrix}$$

This completes the algorithm for the first ($j = 1$) column. Therefore, we move on to the second one, ($j = 2$):

$$r_{22} = \| \mathbf{v}^{(2)} \| = \left\| \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\| = 2 \qquad \Longrightarrow \qquad \mathbf{q}^{(2)} = \frac{1}{2}\mathbf{v}^{(2)} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Then remove any component in the direction of this vector from the remaining ($k = 3$) column,

$$r_{23} = {\mathbf{q}^{(2)}}^H \mathbf{v}^{(3)} = [\ \tfrac{1}{2} \quad -\tfrac{1}{2} \quad \tfrac{1}{2} \quad -\tfrac{1}{2}\ ] \begin{bmatrix} 4 \\ -2 \\ 2 \\ -4 \end{bmatrix} = \frac{12}{2} = 6$$

solution:

yielding

$$\mathbf{v}^{(3)} \;=\; \mathbf{v}^{(3)} - r_{23}\mathbf{q}^{(2)} = \begin{bmatrix} 4 \\ -2 \\ 2 \\ -4 \end{bmatrix} - (6) \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

This completes the algorithm for the second ($j = 2$) column. So we move on to

$$r_{33} = \| \mathbf{v}^{(2)} \| = \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\| = 2 \qquad \Longrightarrow \qquad \mathbf{q}^{(3)} = \frac{1}{2}\mathbf{v}^{(3)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Since there are no more columns to remove components in the direction of $\mathbf{q}^{(3)}$ from, we're done!

The fact that the $\mathbf{q}^{(i)}$ we've found are orthogonormal, i.e. that can easily be checked. For example

$$\mathbf{q}^{(1)^T}\mathbf{q}^{(3)} = [\ \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\ ] \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = 0$$

and

$$\| \mathbf{q}^{(2)} \| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2} = 1$$

Therefore

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

solution:

and, using the values computed above:

$$\mathbf{R} = \begin{bmatrix} 2 & -4 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Direct computation will verify that

$$\mathbf{Q}\,\mathbf{R} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

7. Find a sequence of upper triangular matrices $\tilde{\mathbf{R}}^{(i)}$, each corresponding to a single step of the classic Gram-Schmidt method, such that the matrix $\mathbf{Q}$ in the $\mathbf{QR}$ factorization in problem 5 can be written:

$$\mathbf{Q} = \mathbf{A}\tilde{\mathbf{R}}^{(1)}\tilde{\mathbf{R}}^{(2)}\tilde{\mathbf{R}}^{(3)}$$

Show also by direct computation that the matrix $\mathbf{R}$ from the $\mathbf{QR}$ factorization can be written in terms of the inverses of these $\tilde{\mathbf{R}}^{(i)}$ as:

$$\mathbf{R} = \left(\tilde{\mathbf{R}}^{(3)}\right)^{-1}\left(\tilde{\mathbf{R}}^{(2)}\right)^{-1}\left(\tilde{\mathbf{R}}^{(1)}\right)^{-1}$$

solution:

We have already performed the classic Gram-Schmidt algorithm on this matrix, so we can use any results from that problem here. Specifically, observe the first step there was:

$$\mathbf{q}^{(1)} = \frac{1}{2}\mathbf{a}^{(1)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

But, in block matrix form, we can write this as

$$\left[\mathbf{q}^{(1)} \vdots \mathbf{a}^{(2)} \vdots \mathbf{a}^{(3)}\right] = \left[\mathbf{a}^{(1)} \vdots \mathbf{a}^{(2)} \vdots \mathbf{a}^{(3)}\right] \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{\mathbf{R}}^{(1)}} = \mathbf{A}\,\tilde{\mathbf{R}}^{(1)}$$

In the next step, we effectively first replace $\mathbf{a}^{(2)}$ by

$$\mathbf{v}^{(2)} = \mathbf{a}^{(2)} - (-4)\mathbf{q}^{(1)}$$

then replace that vector (which we interpret as the component of $\mathbf{a}^{(2)}$ orthogonal to $\mathbf{q}^{(1)}$) by

$$\mathbf{q}^{(2)} = \frac{1}{2}\mathbf{v}^{(2)}$$

In block matrix form, this is equivalent to

$$\left[\mathbf{q}^{(1)} \vdots \mathbf{q}^{(2)} \vdots \mathbf{a}^{(3)}\right] = \left[\mathbf{q}^{(1)} \vdots \mathbf{a}^{(2)} \vdots \mathbf{a}^{(3)}\right] \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

solution:

or, equivalently, multiplying the two matrices and using our first step result

$$\left[\mathbf{q}^{(1)} \vdots \mathbf{q}^{(2)} \vdots \mathbf{a}^{(3)}\right] = \underbrace{\left[\mathbf{q}^{(1)} \vdots \mathbf{a}^{(2)} \vdots \mathbf{a}^{(3)}\right]}_{\mathbf{A}\tilde{\mathbf{R}}^{(1)}} \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{\mathbf{R}}^{(2)}} = \mathbf{A}\tilde{\mathbf{R}}^{(1)}\tilde{\mathbf{R}}^{(2)}$$

Finally, we addressed the third column by first replacing $\mathbf{a}^{(3)}$ by

$$\mathbf{v}^{(3)} = \mathbf{a}^{(3)} - (2)\mathbf{q}^{(1)} - (6)\mathbf{q}^{(2)}$$

then finding

$$\mathbf{q}^{(3)} = \frac{1}{2}\mathbf{v}^{(3)}$$

In block matrix form, we can express this as

$$\left[\mathbf{q}^{(1)} \vdots \mathbf{q}^{(2)} \vdots \mathbf{q}^{(3)}\right] = \left[\mathbf{q}^{(1)} \vdots \mathbf{q}^{(2)} \vdots \mathbf{a}^{(3)}\right] \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Mulitplying the two matrices on the right and using our second result yields

$$\underbrace{\left[\mathbf{q}^{(1)} \vdots \mathbf{q}^{(2)} \vdots \mathbf{q}^{(3)}\right]}_{\mathbf{Q}} = \underbrace{\left[\mathbf{q}^{(1)} \vdots \mathbf{q}^{(2)} \vdots \mathbf{a}^{(3)}\right]}_{\mathbf{A}\tilde{\mathbf{R}}^{(1)}\tilde{\mathbf{R}}^{(2)}} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}_{\tilde{\mathbf{R}}^{(3)}} = \mathbf{A}\tilde{\mathbf{R}}^{(1)}\tilde{\mathbf{R}}^{(2)}\tilde{\mathbf{R}}^{(3)}$$

or equivalently, as can be verified by direct computation

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

(Note how each $\tilde{\mathbf{R}}^{(i)}$ has exactly one column that not a column of the identity.)

solution:

Finally, direct computation shows that

$$\left(\tilde{\mathbf{R}}^{(3)}\right)^{-1}\left(\tilde{\mathbf{R}}^{(2)}\right)^{-1}\left(\tilde{\mathbf{R}}^{(1)}\right)^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{R}$$

8. Find a sequence of upper triangular matrices $\mathbf{R}^{(i)}$, each corresponding to a single step of the modified Gram-Schmidt method, such that the matrix $\mathbf{Q}$ in the $\mathbf{QR}$ factorization in problem 6 can be written:

$$\mathbf{Q} = \mathbf{A}\mathbf{R}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}$$

Show also by direct computation that the matrix $\mathbf{R}$ from the $\mathbf{QR}$ factorization can be written in terms of the inverses of these $\mathbf{R}^{(i)}$ as:

$$\mathbf{R} = \left(\mathbf{R}^{(3)}\right)^{-1}\left(\mathbf{R}^{(2)}\right)^{-1}\left(\mathbf{R}^{(1)}\right)^{-1}$$

solution:

We have already performed the modified Gram-Schmidt algorithm on this matrix, so we can use any results from that problem here. Specifically, observe the first step there was:

$$\mathbf{v}^{(i)} = \mathbf{a}^{(i)} \quad , \quad i = 1, 2, 3$$

followed immediately by

$$\mathbf{q}^{(1)} = \frac{1}{2}\mathbf{a}^{(1)}$$

and then removed any components in the direction of $\mathbf{q}^{(1)}$ from the remaining columns, i.e.

$$\begin{aligned} \mathbf{v}^{(2)} &= \mathbf{v}^{(2)} - (-4)\mathbf{q}^{(1)} \quad \text{and} \\ \mathbf{v}^{(3)} &= \mathbf{v}^{(3)} - (2)\mathbf{q}^{(1)} \end{aligned}$$

In block matrix form, we can express this as

$$\left[\mathbf{q}^{(1)} \vdots \mathbf{v}^{(2)} \vdots \mathbf{v}^{(3)}\right] = \left[\mathbf{a}^{(1)} \vdots \mathbf{a}^{(2)} \vdots \mathbf{a}^{(3)}\right] \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or multiplying out the two matrices on the right

$$\left[\mathbf{q}^{(1)} \vdots \mathbf{v}^{(2)} \vdots \mathbf{v}^{(3)}\right] = \underbrace{\left[\mathbf{a}^{(1)} \vdots \mathbf{a}^{(2)} \vdots \mathbf{a}^{(3)}\right]}_{\mathbf{A}} \underbrace{\begin{bmatrix} \frac{1}{2} & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}^{(1)}} = \mathbf{A}\mathbf{R}^{(1)}$$

solution:

In the next step, we first replace $\mathbf{v}^{(2)}$ by

$$\mathbf{q}^{(2)} = \frac{1}{2}\mathbf{v}^{(2)}$$

and then remove any components in the direction of $\mathbf{q}^{(2)}$ from the remaining columns, i.e.

$$\mathbf{v}^{(3)} = \mathbf{v}^{(3)} - (6)\mathbf{q}^{(2)}$$

In block matrix form, this is equivalent to

$$\left[\mathbf{q}^{(1)} \vdots\, \mathbf{q}^{(2)} \vdots\, \mathbf{v}^{(3)}\right] = \left[\mathbf{q}^{(1)} \vdots\, \mathbf{v}^{(2)} \vdots\, \mathbf{v}^{(3)}\right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

(Note this is not, strictly speaking, an equality in the mathematical sense, but in the programming sense, i.e. in the way that, in MATLAB, e.g.

$$x = x + y$$

replaces the value currently stored in the memory location labeled x with the value $x + y$.) But, multiplying out the two matrices on the right and using our earlier result, we have

$$\left[\mathbf{q}^{(1)} \vdots\, \mathbf{q}^{(2)} \vdots\, \mathbf{v}^{(3)}\right] = \underbrace{\left[\mathbf{q}^{(1)} \vdots\, \mathbf{v}^{(2)} \vdots\, \mathbf{v}^{(3)}\right]}_{\mathbf{AR}^{(1)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -3 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}^{(2)}} = \mathbf{AR}^{(1)}\mathbf{R}^{(2)}$$

Finally, we addressed the third column by replacing $\mathbf{v}^{(3)}$ by

$$\mathbf{q}^{(3)} = \frac{1}{2}\mathbf{v}^{(3)}$$

In block matrix form, we can express this as

$$\underbrace{\left[\mathbf{q}^{(1)} \vdots\, \mathbf{q}^{(2)} \vdots\, \mathbf{q}^{(3)}\right]}_{\mathbf{Q}} = \underbrace{\left[\mathbf{q}^{(1)} \vdots\, \mathbf{q}^{(2)} \vdots\, \mathbf{v}^{(3)}\right]}_{\mathbf{AR}^{(1)}\mathbf{R}^{(2)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}_{\mathbf{R}^{(3)}} = \mathbf{AR}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}$$

solution:

Therefore, we have

$$\mathbf{Q} = \mathbf{A}\mathbf{R}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}$$

or equivalently, as direct computation will verify

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

(Note, in contrast to the classical Gram-Schmidt method, each $\mathbf{R}^{(i)}$ here has exactly one row that is not a row of the identity.)

Finally, direct computation also shows that

$$\left(\mathbf{R}^{(3)}\right)^{-1}\left(\mathbf{R}^{(2)}\right)^{-1}\left(\mathbf{R}^{(1)}\right)^{-1} = \begin{bmatrix} 2 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{R}$$

9. Use the modified Gram-Schmidt method to produce a reduced $\mathbf{Q}\,\mathbf{R}$ factorization of

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 6 \\ -1 & -4 \\ 1 & 4 \end{bmatrix}$$

solution:

The modified Gram-Schmidt algorithm, modified to produce a $\mathbf{Q}\,\mathbf{R}$ factorization as a byproduct of producing an orthonormal basis, is:

(1) Let: $\mathbf{v}^{(i)} = \mathbf{a}^{(1)} \quad , i = 1, \ldots, n$

(2) For $j = 1, \ldots, n$:

Form: $\mathbf{q}^{(j)} = r_{jj}\ \mathbf{v}^{(j)} \qquad \Longrightarrow \qquad r_{jj} = \|\, \mathbf{v}^{(j)}\, \|$

Remove the component in the direction of $\mathbf{q}^{(j)}$ from each remaining $\mathbf{v}^{(k)}$ using:

$$\mathbf{v}^{(k)} = \mathbf{v}^{(k)} - r_{jk}\mathbf{q}^{(j)}$$

where $r_{jk} = \mathbf{q}^{(j)^H}\ \mathbf{v}^{(k)}$

Therefore, in this problem

$$\mathbf{v}^{(1)} = \begin{bmatrix} 5 \\ 3 \\ -1 \\ 1 \end{bmatrix} \quad , \text{ and } \quad \mathbf{v}^{(2)} = \begin{bmatrix} 2 \\ 6 \\ -4 \\ 4 \end{bmatrix}$$

So, for $j = 1$, $r_{11} = \|\, \mathbf{v}^{(1)}\, \| = \sqrt{5^2 + 3^2 + (-1)^2 + 1^2} = \sqrt{36} = 6$ and therefore

$$\mathbf{q}^{(1)} = \frac{1}{6}\,\mathbf{v}^{(1)} = \begin{bmatrix} \frac{5}{6} \\ \frac{3}{6} \\ -\frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$$

Now remove any components of this from the remaining columns, i.e. for $k = 2$:

$$r_{12} = \mathbf{q}^{(1)^H}\ \mathbf{v}^{(2)} = [\ \tfrac{5}{6} \quad \tfrac{3}{6} \quad -\tfrac{1}{6} \quad \tfrac{1}{6}\ ] \begin{bmatrix} 2 \\ 6 \\ -4 \\ 4 \end{bmatrix} = \frac{36}{6} = 6$$

solution:

and so $\mathbf{v}^{(2)}$ becomes

$$\mathbf{v}^{(2)} = \begin{bmatrix} 2 \\ 6 \\ -4 \\ 4 \end{bmatrix} - (6) \begin{bmatrix} \frac{5}{6} \\ \frac{3}{6} \\ -\frac{1}{6} \\ \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

This completes the algorithm for the first ($j = 1$) column. Therefore, we move on to the second one, ($j = 2$):

$$r_{22} = \| \mathbf{v}^{(2)} \| = \left\| \begin{bmatrix} -3 \\ 3 \\ -3 \\ 3 \end{bmatrix} \right\| = 6 \qquad \Longrightarrow \qquad \mathbf{q}^{(2)} = \frac{1}{6}\,\mathbf{v}^{(2)} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

This completes the algorithm for the second ($j = 2$) and, in this case, last column. So we're done!

The fact that the $\mathbf{q}^{(i)}$ we've found are orthogonormal can easily be checked and is omitted here. Therefore, using the values computed above:

$$\mathbf{Q} = \begin{bmatrix} \frac{5}{6} & \frac{1}{2} \\ \frac{3}{6} & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} 6 & 6 \\ 0 & 6 \end{bmatrix}$$

Direct computation will verify that

$$\mathbf{Q}\,\mathbf{R} = \begin{bmatrix} \frac{5}{6} & \frac{1}{2} \\ \frac{3}{6} & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 6 \\ -1 & -4 \\ 1 & 4 \end{bmatrix}$$

10. Create an orthogonal plane (Givens) rotation matrix ($\mathbf{Q}$) which uses the third row of:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

to zero out the element currently in the (1,3) position.

solution:

In $\Re^4$, the rotation (Givens) matrix that combines the first and third rows will have the form:

$$\mathbf{Q} = \begin{bmatrix} c & 0 & -s & 0 \\ 0 & 1 & 0 & 0 \\ s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and will produce the result:

$$R_1 \leftarrow cR_1 - sR_3$$
$$R_3 \leftarrow sR_1 + cR_3$$

and therefore

$$a_{13} \leftarrow ca_{13} - sa_{33}$$

which, in order to zero out a_{13} while also ensuring $c^2 + s^2 = 1$ requires

$$c = \frac{a_{33}}{\sqrt{a_{13}^2 + a_{33}^2}} \quad \text{and} \quad s = \frac{a_{13}}{\sqrt{a_{13}^2 + a_{33}^2}}$$

and so

$$c = \frac{3}{\sqrt{4^2 + 3^2}} = \frac{3}{5} \quad \text{and} \quad s = \frac{4}{\sqrt{4^2 + 3^2}} = \frac{4}{5}$$

or

$$\mathbf{Q} = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

solution:

Direct calculation will show:

$$\mathbf{Q}\,\mathbf{A} = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & 0 \\ 1 & -3 & -1 \\ \frac{7}{5} & -\frac{7}{5} & 5 \\ 1 & -3 & -3 \end{bmatrix}$$

Note that the zero is in the correct position, and that, as should have been expected, the second and fourth rows of the product are unchanged from the original matrix.

11. Produce a reflection (Householder) matrix ($\mathbf{Q}$) and the associated vector ($\mathbf{u}$) which will zero out the elements below the *second row* in the first column of:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ -2 & -3 & -1 \\ 4 & -1 & 3 \\ 2 & -3 & -3 \\ 5 & 1 & -1 \end{bmatrix}$$

solution:

Householder reflections have the form:

$$\mathbf{Q} = \mathbf{I} - 2\frac{\mathbf{u}\,\mathbf{u}^T}{\mathbf{u}^T\,\mathbf{u}}$$

in order to specifically zero out the elements in rows three through five of column one, $\mathbf{u}$ must have the general form:

$$\mathbf{u} = \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

where

$$u_i = a_{i1}\ , \quad i = 3,4,5 \quad \text{and} \quad u_2 = a_{21} - \sqrt{a_{21}^2 + a_{31}^2 + a_{41}^2 + a_{51}^2}$$

(where we choose the minus sign because a_{21} is negative). Therefore

$$u_2 = (-2) - \sqrt{(-2)^2 + (4)^2 + (2)^2 + (5)^2} = (-2) - \sqrt{49} = -9$$

and therefore

$$\mathbf{u} = \begin{bmatrix} 0 \\ -9 \\ 4 \\ 2 \\ 5 \end{bmatrix}$$

Given this we can then construct $\mathbf{Q}$ as:

solution:

$$\mathbf{Q} = \mathbf{I} - 2\frac{\mathbf{u}\,\mathbf{u}^T}{\mathbf{u}^T\,\mathbf{u}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ -9 \\ 4 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 0 & -9 & 4 & 2 & 5 \end{bmatrix}}{\begin{bmatrix} 0 & -9 & 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ -9 \\ 4 \\ 2 \\ 5 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{126} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 81 & -36 & -18 & -45 \\ 0 & -36 & 16 & 8 & 20 \\ 0 & -18 & 8 & 4 & 10 \\ 0 & -45 & 20 & 10 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{7} & \frac{4}{7} & \frac{2}{7} & \frac{5}{7} \\ 0 & \frac{4}{7} & \frac{47}{63} & -\frac{8}{63} & -\frac{20}{63} \\ 0 & \frac{2}{7} & -\frac{8}{63} & \frac{59}{63} & -\frac{10}{63} \\ 0 & \frac{5}{7} & -\frac{20}{63} & -\frac{10}{63} & \frac{38}{63} \end{bmatrix}$$

Direct calculation can then show:

$$\mathbf{Q}\,\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{7} & \frac{4}{7} & \frac{2}{7} & \frac{5}{7} \\ 0 & \frac{4}{7} & \frac{47}{63} & -\frac{8}{63} & -\frac{20}{63} \\ 0 & \frac{2}{7} & -\frac{8}{63} & \frac{59}{63} & -\frac{10}{63} \\ 0 & \frac{5}{7} & -\frac{20}{63} & -\frac{10}{63} & \frac{38}{63} \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ -2 & -3 & -1 \\ 4 & -1 & 3 \\ 2 & -3 & -3 \\ 5 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 4 \\ 7 & \frac{1}{7} & \frac{3}{7} \\ 0 & -\frac{151}{63} & \frac{149}{63} \\ 0 & -\frac{233}{63} & -\frac{209}{63} \\ 0 & -\frac{47}{63} & -\frac{113}{63} \end{bmatrix}$$

12. Consider the full **QR** factorization of the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & \frac{3}{6} & -\frac{3}{6} & -\frac{3}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the least squares solution to

$$\mathbf{Ax} = \begin{bmatrix} -43 \\ -3 \\ -10 \\ -16 \end{bmatrix}$$

by using both the normal equations and the **QR** factorization shown. Also confirm that your residual is orthogonal to the column space of **A**

solution:

The normal equations formulation of the least squares problem is:

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$

In this instance that becomes:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & 3 & 3 \end{bmatrix} \begin{bmatrix} -43 \\ -3 \\ -10 \\ -16 \end{bmatrix}$$

or

$$\begin{bmatrix} 4 & 8 \\ 8 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -72 \\ 36 \end{bmatrix}$$

Gaussian elimination produces:

$$\left[\begin{array}{cc|c} 4 & 8 & -72 \\ 8 & 52 & 36 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 4 & 8 & -72 \\ 0 & 36 & 180 \end{array}\right] \Longrightarrow \begin{array}{l} x_1 = -28 \\ x_2 = 5 \end{array}$$

solution:

Given the $\mathbf{QR}$ factorization, the least squares problem can be solved as

$$\mathbf{Ax} = \mathbf{b} \quad \Longrightarrow \quad \mathbf{QRx} = \mathbf{b} \quad \Longrightarrow \quad \mathbf{Rx} = \mathbf{Q}^T\mathbf{b}$$

or in this case:

$$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{6} & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{3}{6} & \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{3}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} -43 \\ -3 \\ -10 \\ -16 \end{bmatrix} = \begin{bmatrix} -36 \\ 30 \\ 3 \\ -3 \end{bmatrix}$$

(Note that when the full matrix $\mathbf{Q}$ is used, the resulting system is, as in this case, almost always inconsistent. This, on reflection, should have been expected.) The consistent part of this system:

$$\begin{array}{rcrcr} 2x_1 & + & 4x_2 & = & -36 \\ & & 6x_2 & = & 30 \end{array} \quad \Longrightarrow \quad \begin{array}{rl} x_1 = & -28 \\ x_2 = & 5 \end{array}$$

i.e. the same solution as obtained by using the normal equations. (A result that should have been expected!) The resulting residual is

$$\mathbf{r} = \mathbf{b} - \mathbf{Ax} = \begin{bmatrix} -43 \\ -3 \\ -10 \\ -16 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -28 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

But then:

$$\mathbf{a}^{(1)^T}\mathbf{r} = [\ 1 \quad 1 \quad 1 \quad 1\,] \begin{bmatrix} 0 \\ 0 \\ 3 \\ -3 \end{bmatrix} = 0$$

and

$$\mathbf{a}^{(2)^T}\mathbf{r} = [\,-3 \quad 5 \quad 3 \quad 3\,] \begin{bmatrix} 0 \\ 0 \\ 3 \\ -3 \end{bmatrix} = 0$$

i.e. the residuals are orthogonal to the columns (and hence the column space) of $\mathbf{A}$.